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Influence of swirl on the stability of a rod in annular leakage flow

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Abstract

The influence of swirl (flow rotation) on the stability of a rod in annular leakage flow is investigated. Under the assumption of laminar flow and plane vibrations (no whirling), it is shown that the swirl acts, in effect, as an elastic foundation with negative foundation stiffness, the magnitude being proportional to the mean circumferential flow rate squared. Consequently, swirl always lowers the critical axial flow speed in case of divergence instability of a rod of finite length. Numerical analysis is needed to predict the effect of swirl in case of flutter instability of a finite rod; this is not performed here. However, for the flutter-like instability of travelling waves in an infinite rod-channel system, it is shown analytically that swirl again always lowers the critical axial flow speed. Finally, it is found that by circumferential flow alone, the travelling waves are extinguished at a certain flow rate, followed by a divergence-like instability.

Keywords: Stability; Leakage-flow; Swirling flow; Divergence; Flutter; Travelling wave

1. Introduction

The effects of swirl on the stability of slender axisymmetric structures (beams and shells) in axial flow through nonnarrow gaps have been studied theoretically by Srinivasan (1971), Lai and Chow (1973), Chen and Bert (1977), Cortelezzi et al. (2004), and experimentally by Dowell et al. (1974).

When the fluid gap is narrow, the flow is often referred to as a leakage-flow. By such flows, both fluid viscosity and inertia effects are of importance, i.e. neither of them can be ignored. Leakage-flow-induced oscillations have received a good deal of attention in recent years, with research being driven mainly, it seems, by 'real world' industrial problems and concerns, often in connection with power-generation (Païdoussis, 2004).

Studies of axisymmetric leakage flow problems, involving a rigid centre-body with one or two degrees of freedom, were carried out by Hobson (1982), Mateescu and Païdoussis (1987), and Li et al. (1998, 2002). Tanaka et al. (2001) considered a system of connected rigid centre-bodies as a model of a high-speed train in a tunnel. Studies involving a flexible centre-body were carried out by Païdoussis and Pettigrew (1979), Païdoussis et al. (1990), Fujita and Shintani (2001), Fujita et al. (2004), and Langthjem et al. (2006).

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In parallel with these studies, problems involving plates in leakage-flow channels have also received a good deal of attention. Rigid plates, involving again one or two degrees of freedom, were considered by Mulcahy (1988) and Inada and Hayama (1990a,b). Inada and Hayama's (1990a) analysis was extended to deal with a flexible plate by Nagakura and Kaneko (1993). Large-amplitude plate vibrations, requiring a nonlinear analysis, were studied by Wu and Kaneko (2005). An elastic sheet subjected to a one-sided leakage flow was studied by Hosoi and Mahadevan (2004), including an investigation of a microscale set-up, where van der Waals (intermolecular) forces also are taken into consideration.

In returning to the axisymmetric configuration, it is noticed that the influence of swirl on the dynamic stability of the central rod has received little, if any, attention, although it is of industrial concern. Against this background, the purpose of this paper is to derive and discuss a few general, analytical results regarding the effects of swirl on the stability of a slender, flexible rod in annular leakage flow.

It is anticipated that most industrial flows are turbulent. It might also be expected that the swirling flow will trigger whirling, and thus three-dimensional, motions of the rod. But turbulence and three-dimensional vibrations will complicate the analysis a great deal. To obtain simple, analytical results the present paper make the assumptions of laminar flow and plane vibrations. Whirling instability should be considered at the next stage. The authors plan to address this problem in a future publication.

2. The governing equations

2.1. Fluid equations

Consider an incompressible, laminar flow through the narrow gap between an outer rigid cylinder (of radius $R + \bar{H}$) and an inner flexible rod (of radius R), described in terms of cylindrical polar coordinates (r, θ, Y). The rod is assumed to vibrate only in the $\theta = 0$ plane . The mean (steady) fluid gap $\bar{H}$ is so narrow, relative to the radius R and the length L of the flexible rod, that the flow there resembles the flow within a boundary layer. This implies that the pressure difference across the gap, in the radial (r) direction, is negligibly small. Also, the effect of curvature can be ignored. The flow is then 'two-dimensional', and can be described in terms of the coordinates (X, Y), where $X = R\theta$, $0 \leqslant \theta \leqslant 2\pi$.

The relations between pressure P, flow rate in circumferential (X) direction Q_X, and flow rate in axial (Y) direction Q_Y, can be expressed as in Inada and Hayama (1990a), Li et al. (1998, 2002), Fujita and Shintani (2001):

$$\frac{1}{\rho}\frac{\partial P}{\partial X} = -\frac{1}{H}\left\{\frac{\partial Q_X}{\partial t} + \frac{\partial}{\partial X}\left(\frac{Q_X^2}{H}\right) + \frac{\partial}{\partial Y}\left(\frac{Q_X Q_Y}{H}\right) + 12\nu\frac{Q_X}{H^2}\right\}, \tag{1}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial Y} = -\frac{1}{H}\left\{\frac{\partial Q_Y}{\partial t} + \frac{\partial}{\partial X}\left(\frac{Q_X Q_Y}{H}\right) + \frac{\partial}{\partial Y}\left(\frac{Q_Y^2}{H}\right) + 12\nu\frac{Q_Y}{H^2}\right\}, \tag{2}$$

where H is the unsteady fluid gap, t is the time, ρ is the fluid density, and ν is the fluid kinematic viscosity. The flow rates Q_X and Q_X are obtained by integrating the flow velocities in the X and Y directions, respectively, over the fluid gap H.

The equation of continuity takes the form

$$\frac{\partial Q_X}{\partial X} + \frac{\partial Q_Y}{\partial Y} + \frac{\partial H}{\partial t} = 0. \tag{3}$$

To investigate the stability of the coupled fluid-structure system with respect to small perturbations, a set of linear relations between pressure and flow rates is derived from (1), (2), and (3). To this end the variables (fluid gap H, pressure P, and flow rates Q_X and Q_Y) are separated into steady and unsteady (disturbance, or perturbation) parts (Inada and Hayama, 1990a):

$$\begin{aligned} &H(X,Y,t) = \bar{H} + \Delta H(X,Y,t), \quad P(X,Y,t) = \bar{P} + \Delta P(X,Y,t), \\ &Q_X(X,Y,t) = \bar{Q}_X + \Delta Q_X(X,Y,t), \quad Q_Y(X,Y,t) = \bar{Q}_Y + \Delta Q_Y(X,Y,t). \end{aligned} \tag{4}$$

Here the quantities with an overbar are steady, and quantities with a Δ are unsteady. As indicated in the equations, the steady quantities are assumed to be constants. Without swirl, $\bar{Q}_X = 0$; this is the case considered in earlier works (Li et al., 1998, 2002; Fujita and Shintani, 2001).

The continuity equation for the perturbations takes the form

$$\frac{\partial \Delta Q_X}{\partial X} + \frac{\partial \Delta Q_Y}{\partial Y} + \frac{\partial \Delta H}{\partial t} = 0. \tag{5}$$

Utilizing this equation, the linearized unsteady parts of (1) and (2) can be written as

$$\frac{1}{\rho}\frac{\partial \Delta P}{\partial X} = -\frac{1}{\bar{H}}\frac{\partial \Delta Q_X}{\partial t} - \frac{\bar{Q}_X}{\bar{H}^2}\frac{\partial \Delta Q_X}{\partial X} - \frac{\bar{Q}_Y}{\bar{H}^2}\frac{\partial \Delta Q_X}{\partial Y} + \frac{\bar{Q}_X}{\bar{H}^2}\frac{\partial \Delta H}{\partial t} + \frac{\bar{Q}_X^2}{\bar{H}^3}\frac{\partial \Delta H}{\partial X} + \frac{\bar{Q}_X\bar{Q}_Y}{\bar{H}^3}\frac{\partial \Delta H}{\partial Y} - 12\nu\frac{\Delta Q_X}{\bar{H}^3} + 36\nu\frac{\bar{Q}_X}{\bar{H}^4}\Delta H, \tag{6}$$

$$\frac{1}{\rho}\frac{\partial \Delta P}{\partial Y} = -\frac{1}{\bar{H}}\frac{\partial \Delta Q_Y}{\partial t} - \frac{\bar{Q}_X}{\bar{H}^2}\frac{\partial \Delta Q_Y}{\partial X} - \frac{\bar{Q}_Y}{\bar{H}^2}\frac{\partial \Delta Q_Y}{\partial Y} + \frac{\bar{Q}_Y}{\bar{H}^2}\frac{\partial \Delta H}{\partial t} + \frac{\bar{Q}_X\bar{Q}_Y}{\bar{H}^3}\frac{\partial \Delta H}{\partial X} + \frac{\bar{Q}_Y^2}{\bar{H}^3}\frac{\partial \Delta H}{\partial Y} - 12\nu\frac{\Delta Q_Y}{\bar{H}^3} + 36\nu\frac{\bar{Q}_Y}{\bar{H}^4}\Delta H. \tag{7}$$

These equations differ from those derived in the just-mentioned earlier works only by the appearance of terms involving $\bar{Q}_X$.

Next, differentiate (6) with respect to X and (7) with respect to Y, and add the two resulting equations; this gives

$$\frac{1}{\rho}\left(\frac{\partial^2 \Delta P}{\partial X^2} + \frac{\partial^2 \Delta P}{\partial Y^2}\right) = \frac{1}{\bar{H}}\frac{\partial^2 \Delta H}{\partial t^2} + \frac{12\nu}{\bar{H}^3}\frac{\partial \Delta H}{\partial t} + 2\frac{\bar{Q}_X}{\bar{H}^2}\frac{\partial^2 \Delta H}{\partial t \partial X} + 2\frac{\bar{Q}_Y}{\bar{H}^2}\frac{\partial^2 \Delta H}{\partial t \partial Y} + \frac{\bar{Q}_X^2}{\bar{H}^3}\frac{\partial^2 \Delta H}{\partial X^2} + \frac{\bar{Q}_Y^2}{\bar{H}^3}\frac{\partial^2 \Delta H}{\partial Y^2}$$
$$+ 2\frac{\bar{Q}_X\bar{Q}_Y}{\bar{H}^3}\frac{\partial^2 \Delta H}{\partial X \partial Y} + \frac{36\nu}{\bar{H}^4}\left(\bar{Q}_X\frac{\partial \Delta H}{\partial X} + \bar{Q}_Y\frac{\partial \Delta H}{\partial Y}\right), \tag{8}$$

a relation involving only the two perturbations ΔH and ΔP.

Let $\Delta H_0(Y,t)$ denote the deflection of the rod at position Y, in the vibrational plane $\theta = 0$. The deflection in the azimuthal direction $\theta = X/R$ is then given by $\Delta H_0(Y,t)\cos\theta$; see Fig. 1. Due to the narrow gap geometry, and to the assumption of small-amplitude vibrations, the fluid gap and fluid pressure therein can be assumed to vary with θ in the same way (Hobson, 1982; Païdoussis, 2004), i.e.,

$$\Delta H(X,Y,t) = \Delta H_0(Y,t)\cos\theta, \quad \Delta P(X,Y,t) = \Delta P_0(Y,t)\cos\theta. \tag{9}$$

The fluid force acting on the rod in the vibrational plane $\theta = 0$ is obtained by integrating, over the rod surface, the projection of the pressure ΔP onto the plane $\theta = 0$:

$$\Delta F(Y,t) = \int_0^{2\pi} \Delta P R \cos\theta \, d\theta. \tag{10}$$

Integrating (8) in this way (i.e., multiplication by $R\cos\theta$, followed by integration over θ) gives, with application of (9),

$$\frac{1}{\rho\pi R}\left(-\frac{1}{R^2}\Delta F + \frac{\partial^2 \Delta F}{\partial Y^2}\right) = \frac{1}{\bar{H}}\frac{\partial^2 \Delta H_0}{\partial t^2} + \frac{12\nu}{\bar{H}^3}\frac{\partial \Delta H_0}{\partial t} + 2\frac{\bar{Q}_Y}{\bar{H}^2}\frac{\partial^2 \Delta H_0}{\partial t \partial Y}$$
$$-\frac{1}{R^2}\frac{\bar{Q}_X^2}{\bar{H}^3}\Delta H_0 + \frac{\bar{Q}_Y^2}{\bar{H}^3}\frac{\partial^2 \Delta H_0}{\partial Y^2} + 36\nu\frac{\bar{Q}_Y}{\bar{H}^4}\frac{\partial \Delta H_0}{\partial Y}. \tag{11}$$

The great simplification from (8) is due to the disappearance, by integration, of the terms differentiated once with respect to X.

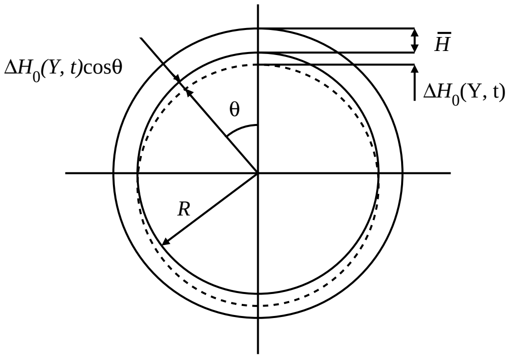

Fig. 1. Steady and unsteady components of the fluid gap.

2.2. Rod equation

The (Bernoulli–Euler type) equation of motion for a slender rod with Kelvin–Voigt-type internal damping, and subjected to a distributed load $\Delta F(Y,t)$, is given by (Païdoussis, 1998)

$$M\frac{\partial^2 \Delta H_0}{\partial t^2} + E^* I\frac{\partial^5 \Delta H_0}{\partial t \partial Y^4} + EI\frac{\partial^4 \Delta H_0}{\partial Y^4} = \Delta F, \tag{12}$$

where M is the mass per unit length, E^* is the coefficient of viscoelastic damping, E is the modulus of elasticity, and I is the area moment of inertia.

2.3. Nondimensionalization

By utilizing the nondimensional variables and parameters

$$\begin{aligned} &x=\theta=\frac{X}{R},\quad y=\frac{Y}{L},\quad h=\frac{\Delta H_0}{\bar{H}},\quad \varepsilon=\frac{R}{L},\quad \tau=\frac{t}{L^2}\left(\frac{EI}{M}\right)^{1/2},\\ &f=\Delta F\frac{L^4}{\bar{H}EI},\quad \beta=12v\left(\frac{L}{\bar{H}}\right)^2\left(\frac{M}{EI}\right)^{1/2},\\ &\bar{q}_x=\bar{Q}_X\frac{L}{\bar{H}}\left(\frac{M}{EI}\right)^{1/2},\quad \bar{q}_y=\bar{Q}_Y\frac{L}{\bar{H}}\left(\frac{M}{EI}\right)^{1/2}, \end{aligned}\tag{13}$$

Eq. (11) can be expressed in nondimensional form as

$$\frac{1}{\kappa}\left(-f+\varepsilon^2\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^2 h}{\partial \tau^2}+\beta\frac{\partial h}{\partial \tau}+2\bar{q}_y\frac{\partial^2 h}{\partial \tau\partial y}-\frac{\bar{q}_x^2}{\varepsilon^2}h+\bar{q}_y^2\frac{\partial^2 h}{\partial y^2}+3\beta\bar{q}_y\frac{\partial h}{\partial y}, \tag{14}$$

where

$$\kappa=\frac{\rho\pi R^3}{\bar{H}M}. \tag{15}$$

The nondimensionalized version of (12) is

$$\frac{\partial^2 h}{\partial \tau^2}+c\frac{\partial^5 h}{\partial \tau\partial y^4}+\frac{\partial^4 h}{\partial y^4}=f. \tag{16}$$

2.4. Coupling of fluid and structure governing equations

Combine, finally, Eqs. (14) and (16), through elimination of f, to give

$$\begin{aligned} &\frac{\partial^2}{\partial \tau^2}\left\{(1+\kappa)h-\varepsilon^2\frac{\partial^2 h}{\partial y^2}\right\}+\frac{\partial}{\partial \tau}\left\{c\left(\frac{\partial^4 h}{\partial y^4}-\varepsilon^2\frac{\partial^6 h}{\partial y^6}\right)+\kappa\beta h+2\kappa\bar{q}_y\frac{\partial h}{\partial y}\right\}\\ &\quad+\frac{\partial^4 h}{\partial y^4}-\varepsilon^2\frac{\partial^6 h}{\partial y^6}-\kappa\frac{\bar{q}_x^2}{\varepsilon^2}h+\kappa\bar{q}_y^2\frac{\partial^2 h}{\partial y^2}+3\kappa\beta\bar{q}_y\frac{\partial h}{\partial y}=0. \end{aligned}\tag{17}$$

This equation shows explicitly that the circumferential flow ($\bar{q}_x$) acts as an elastic foundation (continuously and uniformly distributed springs) with negative foundation stiffness.

2.5. Boundary conditions

Eq. (17) is, with the solution assumption $h(y,\tau)=\exp(\lambda\tau)\hat{h}(y)$, an ordinary differential equation of sixth order, demanding six boundary conditions for a unique solution. The support conditions for the rod provide four. The remaining two must be related to the fluid.

Inada and Hayama (1990a) formulated fluid boundary conditions which relate the pressure perturbation p to the axial flow-rate perturbation q_y and the rod deflection h. As the flow rate perturbations have been eliminated from (14), such boundary conditions cannot be evaluated, except in the 'trivial' cases with supported ends, where these conditions are identically zero, and thus 'automatically' satisfied.

For geometries where the Bernoulli–Euler beam theory is legitimate, $\varepsilon = R/L = \mathcal{O}(\frac{1}{10})$. It is then reasonable to discard terms multiplied by ε^2. Doing this, Eq. (17) is reduced to an equation of fourth order, and the four structural boundary conditions suffice to determine a unique solution. But the pressure loss associated with a free end of the rod cannot be accounted for by this approach. [This does however not affect the cases studied in the present paper.]

3. Energy balance

3.1. Divergence instability

Divergence is independent of time, and is governed by the terms in the last line of (17). An energy balance equation can be obtained by multiplying these terms by the deflection h(y), followed by integration over the rod (from $y = 0$ to $y = 1$). The (stiffness-related) term multiplied by ε^2 will be ignored, as mentioned above. Also, for the sake of a simple illustration, we will here consider only pinned–pinned and clamped–clamped boundary conditions.[1] In the first case,

$$h|_{y=0} = h|_{y=1} = 0, \quad \left.\frac{\partial^2 h}{\partial y^2}\right|_{y=0} = \left.\frac{\partial^2 h}{\partial y^2}\right|_{y=1} = 0; \tag{18}$$

while in the second case,

$$h|_{y=0} = h|_{y=1} = 0, \quad \left.\frac{\partial h}{\partial y}\right|_{y=0} = \left.\frac{\partial h}{\partial y}\right|_{y=1} = 0. \tag{19}$$

Divergence sets in when the total potential energy $\mathscr{V}$ of the system fails being positive definite. For both sets of boundary conditions the divergence criterion (or critical state condition) becomes

$$\mathscr{V} = \mathscr{U} - (\mathscr{W}_x + \mathscr{W}_y) = 0, \tag{20}$$

with

$$\mathscr{U} = \frac{1}{2}\int_0^1 \left(\frac{\partial^2 h}{\partial y^2}\right)^2 dy,$$
$$\mathscr{W}_x = \frac{1}{2}\frac{\kappa}{\varepsilon^2}\bar{q}_x^2 \int_0^1 h^2\, dy, \quad \mathscr{W}_y = \frac{1}{2}\kappa \bar{q}_y^2 \int_0^1 \left(\frac{\partial h}{\partial y}\right)^2 dy. \tag{21}$$

Here $\mathscr{U}$ is the elastic (bending) energy of the rod, and $\mathscr{W}_x$ and $\mathscr{W}_y$ is the work done by the circumferential and axial flow forces, respectively. All three terms are positive definite. Eqs. (20) and (21) show that the circumferential flow acts in a way similar to the axial flow (i.e., always destabilizing), and that the former alone (i.e., $\bar{q}_x > 0$, $\bar{q}_y = 0$) can initiate divergence.

3.2. Flutter instability

Benjamin (1961) showed, for a system of two articulated fluid-conveying pipes, that flutter is initiated when the energy delivered to the structure by nonconservative (circulatory) fluid forces over one oscillation period exceeds the energy 'drained' by dissipative (damping) forces during that same period. A similar flutter criterion can be written down for the present system. Here the nonconservative fluid forces are associated with the term $3\kappa\beta\bar{q}_y \partial h/\partial y$ in (17), which acts as a uniformly distributed tangential 'follower' force. [It is shown in Langthjem et al. (2006) that flutter instability is possible with both rod ends supported, and that the flutter is a downstream travelling wave.] As the circumferential flow-induced 'elastic foundation' involves only conservative forces, as shown in (20) and (21), it will not enter directly in an energy-based flutter criterion. However, as dissipative forces are present, the circumferential flow will affect the vibration modes, and thus indirectly affect the balance between nonconservative energy input and

[1] It will be seen that this does not affect the conclusions regarding the effect of the circumferential flow in any way, as the corresponding energy term is not modified by the boundary conditions. These conclusions are also not affected by ignoring the term multiplied by ε^2.

dissipation.[2] It does not seem possible, then, to establish a simple analytical criterion, like (20), for the influence of circumferential flow on the flutter boundary. Numerical analysis is needed, and circumferential flow may well be stabilizing in some parameter ranges, and destabilizing in others. The problem will not be considered further in this paper.

4. The stability of travelling waves in a long rod-channel system

4.1. The general case

It may be of interest also to consider the effect of swirl on the stability of waves in a long (infinite) rod-channel system. As the length L then is meaningless, we replace L by the rod radius R in the nondimensional parameters (13). The parameter ε accordingly takes the value of 1.

It is assumed that the rod displacement $h(y,\tau)$ can be represented by the Fourier integral

$$h(y,\tau) = \int_{-\infty}^{\infty} A(k)\,\exp(iky - i\omega\tau)dk, \tag{22}$$

corresponding to a superposition of travelling waves. The single wave component

$$h_k(y,\tau) = A(k)\,\exp(iky - i\omega\tau), \tag{23}$$

having wave number k, is then governed by the equation (dispersion relation)

$$-\omega^2(1+\kappa+k^2) + \omega[2k\kappa\bar{q}_y - i\{ck^4(1+k^2)+\kappa\beta\}] + k^4(1+k^2) - \kappa\bar{q}_x^2 - k^2\kappa\bar{q}_y^2 + i3k\kappa\beta\bar{q}_y = 0. \tag{24}$$

Solving this equation with respect to ω gives

$$\omega = \frac{1}{2\hat{a}}\left\{-\hat{b} \pm \left(\hat{b}^2 - 4\hat{a}\hat{c}\right)^{1/2}\right\}, \tag{25}$$

with

$$\hat{a} = -(1+\kappa+k^2), \quad \hat{b} = 2k\kappa\bar{q}_y - i\{ck^4(1+k^2)+\kappa\beta\},$$
$$\hat{c} = k^4(1+k^2) - \kappa\bar{q}_x^2 - k^2\kappa\bar{q}_y^2 + i3k\kappa\beta\bar{q}_y. \tag{26}$$

For the particular wave component in focus (with wavenumber k) it is seen that the solution ω for which the real part is positive corresponds to a forward (downstream) travelling wave; the solution for which the real part is negative corresponds to a backward (upstream) travelling wave. Instability sets in (the amplitude will grow 'unbounded') if the imaginary part of ω becomes positive. With the real part $\mathscr{Re}(\omega)\neq 0$, this instability is equivalent to flutter in a finite system.

4.2. Case of undamped rod in inviscid fluid flow

If, for the moment, we consider an undamped rod ($c = 0$) in an inviscid fluid flow ($\beta = 0$), an instability criterion (related to a certain wavenumber) can easily be obtained, because the coefficients $\hat{a}$, $\hat{b}$, and $\hat{c}$ are then all real. Instability will set in when $\hat{b}^2 - 4\hat{a}\hat{c} < 0$. Inserting the coefficients (26) into this inequality gives that a wave with wavenumber k is unstable when

$$(1+\kappa+k^2)k^4(1+k^2) < (1+\kappa+k^2)\kappa\bar{q}_x^2 + k^2(1+k^2)\kappa\bar{q}_y^2. \tag{27}$$

Any wavenumber k is present in an infinite rod, as expressed by (22). When $k \to 0$, (27) shows that any nonzero value of either the circumferential flow rate $\bar{q}_x$ or the axial flow rate $\bar{q}_y$ will result in instability. Stable waves are possible if, for example, the rod is supported by an elastic foundation, which can be represented mathematically by replacing $\kappa\bar{q}_x^2$ by $\kappa\bar{q}_x^2 - \eta$, where η is the nondimensional foundation stiffness.

The critical flow rate couple $(\bar{q}_x^2, \bar{q}_y^2)_{crit}$ in the presence of an elastic foundation can be determined as follows: first the value of k which corresponds to the minimum of the discriminant $D = \hat{b}^2 - 4\hat{a}\hat{c}$ must be determined. This value is then

[2] In a nondissipative, circulatory system, the flutter boundary will not be affected by an elastic foundation if the stiffness distribution of the foundation is similar to the mass distribution (Smith and Herrmann, 1972; Sundararajan, 1974). See also Elishakoff (2005) for a recent, comprehensive review of this topic.

to be inserted into the equation $D = 0$, wherefrom $(\bar{q}_x^2, \bar{q}_y^2)_{crit}$ can be determined (Roth, 1964). Analytical determination of the k-value which corresponds to the minimum of D is not trivial, as D is a fourth-order polynomial in k^2, and it will not be pursued further here.

4.3. The effect of swirl by a real (damped) rod in a real (viscous) fluid flow

When structural damping and fluid viscosity are taken into account ($c>0, \beta>0$) the coefficients $\hat{b}$ and $\hat{c}$ are complex, and a simple instability criterion like (27) cannot be derived. If the axial flow is considered as the 'main' flow, it is however possible to show analytically that the circumferential flow always acts to destabilize the system.

This is done by showing that the imaginary part of the derivative $\partial\omega/\partial\bar{q}_x$, evaluated at the onset of instability ($\mathscr{Im}(\omega) = 0$) always is positive. This means that an increase in $\bar{q}_x$ always will result in a lower critical value of $\bar{q}_y$.

Differentiating (25) with respect to $\bar{q}_x$ gives

$$\frac{\partial\omega}{\partial\bar{q}_x} = \pm\frac{2\kappa\bar{q}_x\,\mathrm{sgn}(\bar{q}_x)}{(\hat{b}^2 - 4\hat{a}\hat{c})^{1/2}}. \tag{28}$$

At the instability onset, where $\mathscr{Im}(\omega) = 0$, we have from (25) that $\mathscr{Im}\{\pm(\hat{b}^2 - 4\hat{a}\hat{c})^{\frac{1}{2}}\} = \mathscr{Im}(\hat{b})$. Then,

$$\mathscr{Im}\left[\pm\frac{1}{(\hat{b}^2 - 4\hat{a}\hat{c})^{1/2}}\right] = \mathscr{Im}\left[\frac{1}{2k\kappa\bar{q}_y - i\{ck^4(1+k^2)+\kappa\beta\}}\right] = \frac{ck^4(1+k^2)+\kappa\beta}{\{2k\kappa\bar{q}_y\}^2 + \{ck^4(1+k^2)+\kappa\beta\}^2}. \tag{29}$$

Inserting this result into (28) gives that

$$\mathscr{Im}\left.\frac{\partial\omega}{\partial\bar{q}_x}\right|_{\mathscr{Im}\,\omega=0} = \frac{2\kappa\bar{q}_x\mathrm{sgn}(\bar{q}_x)\{ck^4(1+k^2)+\kappa\beta\}}{\{2k\kappa\bar{q}_y\}^2 + \{ck^4(1+k^2)+\kappa\beta\}^2} > 0, \tag{30}$$

as should be demonstrated.

4.4. Case of circumferential flow alone

We will finally consider the effect of circumferential flow alone, i. e., the axial flow rate $\bar{q}_y = 0$. In this case, Eq. (25) can be written as

$$\omega = \frac{1}{2z_0}\left[\pm\{z_1 - (z_2 + z_3^2)\}^{1/2} - iz_3\right], \tag{31}$$

with

$$z_0 = 1+\kappa+k^2, \quad z_1 = 4(1+\kappa+k^2)k^4(1+k^2) \quad z_2 = 4(1+\kappa+k^2)\kappa\bar{q}_x^2, \quad z_3 = ck^4(1+k^2)+\kappa\beta. \tag{32}$$

At sufficiently small values of $\bar{q}_x$, $z_1 > z_2 + z_3^2$, assuming that c and β are small numbers, and that $k \neq 0$. The quantity $\{z_1 - (z_2 + z_3^2)\}^{1/2}$ will then be a real number. The "+" and the "−" solution correspond to a downstream and an upstream travelling wave, respectively, and as $z_3 > 0$, they are both asymptotically stable.

When $\bar{q}_x$ is increased up to a certain level, the equality $z_1 = z_2 + z_3^2$ will be fulfilled, and neither wave (of that particular wave number k) can exist. When $\bar{q}_x$ is increased further from this point, both roots of (31) will be purely imaginary, and negative.

The instability limit is reached when z_2 becomes equal to z_1. By continued increase, $z_1 - z_2$ will become smaller than zero, and one of the purely imaginary roots will become positive, initiating a divergence-type of instability. This instability will appear to come 'out of the blue', because in the range $z_1 - z_3^2 < z_2 < z_1$ no waves (again, of that particular wavenumber k) exist.

Inserting the coefficients (32) in the 'instability condition' $z_1 = z_2$ gives the critical flow rate as $\bar{q}_x^2 = k^4(1+k^2)/\kappa$, showing that any nonzero circumferential flow rate will initiate instability for $k \to 0$. If the rod is supported by an elastic foundation with nondimensional stiffness η, as considered also in Section 4.2, the critical flow rate is found to be $(\bar{q}_x)_{crit} = (\eta/\kappa)^{1/2}$.

5. Conclusions

A number of results related to the effect of swirl by annular leakage flow have been obtained. The main findings can be summarized as follows.

(1) Swirl acts, in effect, as an elastic foundation with negative foundation stiffness, the magnitude being proportional to the mean circumferential flow rate squared. Accordingly, swirl acts to destabilize the system in the case of divergence instability of a finite rod, i.e. swirl lowers the critical value of the axial flow rate.
(2) Swirl is also destabilizing in the case of a flutter-type of instability of an infinitely long rod, i.e. swirl again lowers the critical value of the axial flow rate.
(3) In an infinitely long rod with circumferential flow alone (no axial flow), the travelling waves are extinguished at a certain flow rate, followed by a divergence-type of instability.

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